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The influence of jet flow on jet noise. Part 1. The noise of unheated jets

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The present paper and part 2 (adjacent) study the sound field produced by a convected point quadrupole embedded in and moving along the axis of a round plug-flow jet. Only subsonic eddy convection velocities are considered. We examine cold jets here and hot jets in part 2. A principal feature of the study is extensive comparison with jet-noise data. It appears that this simple model problem succeeds in explaining all the major interesting features of jet-noise data, on both hot and cold jets, for jet exit velocities in the low supersonic range. Particular success is achieved in explaining aspects of the data not explainable by the Lighthill acoustic-analogy approach. The picture of jet-noise generation that emerges (at least for jet velocities in the low supersonic regime) is in many respects a striking reaffirmation of the Lighthill point of view. It appears that there is an intrinsic or universal distribution of compact quadrupoles, whose strength and frequency distribution scale with the jet velocity and nozzle diameter as would be expected from simple dimensional reasoning, responsible for jet-noise generation. These quadrupoles are of course convected by the mean flow and satisfactory agreement with the data is obtained by assuming that they are devoid of any intrinsic directionality. There appears to be no significant jet Mach number (compressibility) or jet temperature effect on the scaling of this intrinsic distribution. The essential improvement over the Lighthill analysis is the incorporation of mean-flow shrouding effects on the radiation of the convected quadrupoles. It is perhaps no exaggeration to claim that, with the incorporation of such a shrouding effect, the problem of scaling jet noise with regard to the jet velocity, jet temperature, jet size and the angle from the jet axis appears to be completely resolved. (The 'scaling' principle cannot of course be very simply expressed and in fact needs calculations of the sort contained in the present paper to implement it.)

1. Introduction

The generation of aerodynamic noise by free turbulence received its first quantitative formulation in the papers of Lighthill (1952, 1954). The principal quantitative step achieved by Lighthill was his rearrangement of the continuity and momentum equations to yield

$$\partial^2\rho/\partial t^2 - a_0^2\nabla^2\rho = \partial^2 T_{ij}/\partial x_i \partial x_j \ldots, \tag{1}$$

where $T_{ij} = \rho u_i u_j + p_{ij} - a_0^2 \rho \delta_{ij}$. ρ denotes the density, a_0 a reference speed of sound, $\{p_{ij}\}$ the compressive stress tensor and u_i denotes a velocity component. δ_{ij} is the Kronecker delta function. Even if we assume that p_{ij} can be approximated by $p\delta_{ij}$, where p is the pressure, there are two major problems in the use of (1). First, even in cases where the pressure and density are related in an isentropic fashion, for a thermally stratified flow such as a hot jet the relation $dp = a^2 d\rho$ would require employing different values of a^2 in the different regions of the flow (i.e. no one value of a_0^2 could be used to eliminate the source term $p_{ij} - a_0^2 \rho \delta_{ij}$). Second, even if the part $p_{ij} - a_0^2 \rho \delta_{ij}$ of T_{ij} is ignored (as is reasonable for cold jets), the equation

$$\frac{\partial^2 \rho}{\partial t^2} - a_0^2 \nabla^2 \rho = \frac{\partial^2}{\partial x_i \partial x_j} (\rho u_i u_j) \ldots \tag{2}$$

can be used only to derive an integral equation for ρ (assuming the $u_i u_j$ are known). This difficulty was circumvented by Lighthill (1952, 1954) by his approximation of $\rho u_i u_j$ on the right-hand side of (2) by $\rho_j u_i u_j$, where ρ_j is the mean jet density.

On the basis of (2) (with $\rho u_i u_j$ approximated by $\rho_j u_i u_j$), Lighthill produced his famous intensity law (1962), to wit that the far-field intensity varies as

$$I \sim \frac{\rho_j^2 V_j^8 D^2 \mathscr{F}(\theta)}{\rho_0 a_0^5 R^2 (1 - M_c \cos\theta)^5} \cdots . \tag{3}$$

Here V_j denotes the jet exit velocity, D the nozzle diameter, R the radius of measurement, ρ_0 the ambient density, a_0 the ambient speed of sound, M_c the eddy convection Mach number (usually taken as about $0{\cdot}65 V_j/a_0$) and θ the angle from the jet axis where the intensity is measured. $\mathscr{F}(\theta)$ would be the 'intrinsic' directionality factor for the quadrupoles. Lush (1971) concluded by an examination of the overall intensity patterns at low jet velocities (where convection effects are small) that $\mathscr{F}(\theta)$ was approximately unity, i.e. independent of θ. In what follows we shall also assume $\mathscr{F}(\theta)$ to be independent of θ. The theory is of course valid only for $M_c < 1$ (subsonic eddy convection speeds). As Ahuja & Bushell (1973) have pointed out, for cold jets (when $\rho_j \sim \rho_0$), (3) actually yields a potentially extremely valuable scaling principle. It suggests that, if the far-field sound pressure level SPL at a particular frequency f and angle from jet axis θ is analysed by plotting the quantity

$$Q \equiv \mathrm{SPL} - 10 \log (V_j/V_{\mathrm{ref}})^8 - 10 \log (D/R)^2 - 10 \log [(1 - M_c \cos\theta)^{-5}]$$

(where V_{ref} denotes a reference velocity) against a frequency parameter

$$P \equiv fD/V_j(1 - M_c \cos\theta)$$

(this factor denoting a source Strouhal number allowing for a Doppler shift of the source frequency), one should, according to the Lighthill analysis, obtain a universal curve of Q *vs.* P regardless of the jet velocity, frequency or angle of observation, nozzle size or radius of measurement. To account for heating effects the parameter Q should be modified to

$$Q - 10\log(\rho_j^2/\rho_{\mathrm{ref}}^2)$$

(where ρ_{ref} denotes a reference density).

Recent careful experimental studies by Lush (1971), Ahuja & Bushell (1973) and Hoch *et al.* (1972) have shown that such a scaling principle is not able to collapse the data onto one universal curve. For cold jets, it is found that for low values of P an expression of the form $(1 - M_c \cos\theta)^{-5}$ in (3) underestimates the variation of noise with angle. Conversely, at high values of P a factor of the form $(1 - M_c \cos\theta)^{-5}$ overestimates the variation with angle. For hot jets, the data of Hoch *et al.* (1972) appears to be best correlated by replacing ρ_j^2 in (3) with ρ_j^ω, where ω itself is dependent on V_j/a_0, θ and P. It is only for values of V_j/a_0 in excess of about 1·3 that ω approaches 2. Also the effects of heating generally cause the relative spectrum of the far-field pressure to be progressively biased towards the lower frequencies.

The principal contention of this part of the present study is that for unheated jets most of these discrepancies can be resolved if, while retaining the Lighthill notion of ascribing jet noise to convected quadrupoles, we account for the fact that the eddies do not communicate directly with the ambient atmosphere but are subject to a shrouding effect of the mean jet flow. Mathematically this entails further manipulations of (1) to extract explicitly the influence of the mean flow and arrive at an equation which is clearly in the form of an inhomogeneous wave equation driven by convected, solenoidal, turbulent velocity fluctuations. Such a development was originated by Phillips (1960) and has been developed more fully by Lilley (1972) and Goldstein & Howes (1973). Historically the Lilley–Goldstein development appears to have been first anticipated by White in an appendix to Eldred *et al.* (1963). The present study may be regarded as a simplified attempt to solve equations of the type derived by Lilley and Goldstein & Howes where, in the interest of obtaining closed-form solutions and motivated by the desire to avoid obscuring the physics by complicated numerical approaches, the jet flow is modelled as a simple, round, plug-flow jet. The reader's attention is also drawn to the work of Tester & Burrin (1974) and Berman (1974), who have expounded ideas very similar to those contained in the present study. The papers of Ribner (1960, 1962), Powell (1960) and Csanady (1966) also deserve mention as having drawn attention to mean-flow shrouding effects.

It is appropriate at this stage to point out that it is still a matter of substantial controversy as to how relevant the equation developed by Lilley (1972) (the homogeneous form of which is identical to the compressible Orr–Sommerfeld equation employed in the study of stability of parallel shear flows) is to the problem of noise from high-speed jets. In a recent series of lectures, Ffowcs Williams (1975) has explained this point of view in some detail. One way of looking at Lighthill's acoustic-analogy approach would be as follows. Linear source-free acoustics of a uniform stationary medium are governed by the acoustic equation

$$\rho_{tt} - a_0^2 \nabla^2 \rho = \mathscr{L}_0 \rho = 0.$$

Linear acoustics of the same medium including a source distribution $s(\bar{x}, t)$ are governed by $\mathscr{L}_0 \rho = s(\bar{x}, t)$. This means that the equivalent sources in an analogy with the acoustics of a stationary uniform medium are determined by the extent to which ρ_{tt} is not fully balanced by $a_0^2 \nabla^2 \rho$. By his ingenious manipulation of the equations of motion, Lighthill showed that this imbalance was exactly

$$\partial^2 T_{ij} / \partial x_i \partial x_j.$$

The studies of Lilley etc. may be viewed similarly as follows. The equation governing the linear propagation of sound in a parallel shear flow is the Orr–Sommerfeld equation (OS = 0). One may then say that Lilley etc. sought to develop an analogy with linear propagation of sound in a parallel shear flow by seeking to determine by use of the full equations of motion and energy the extent to which OS $\neq$ 0 and calling this the source term. As Ffowcs Williams (1975) has explained, several objections can be raised to this approach. The most serious of these objections is as follows. An exemplary feature of $\mathscr{L}_0 \rho = s(\bar{x}, t)$ is that the only unique solution to this equation satisfying the radiation condition of outgoing waves when $s(\bar{x}, t) \equiv 0$ is $\rho \equiv 0$. In other words, Lighthill's equation has the appealingly self-consistent aspect that no sources imply no sound. This, unfortunately, is not true of Lilley's equation because the homogeneous equation OS = 0 does admit non-trivial time-dependent unstable wave solutions leading to acoustic radiation in the far field. Other less serious objections to Lilley's equations have also been explained by Ffowcs Williams (1975). Analytically it is very difficult to solve in closed form the equation OS = $s(\bar{x}, t)$. One has, in practice, to resort to numerical techniques (see, for example, Tester & Burrin 1974; Berman 1974). Second, the rationale behind the Lilley approach based on the OS equation is to some extent based on regarding the events leading to jet noise as small departures from a basically steady, laminar, parallel shear flow. With occasional flow reversal and local fluctuating velocities sometimes as high as 40% of the local steady velocity, such a view is clearly difficult to defend. Finally, the effect of attempting to calculate the degree to which OS $\neq$ 0 generally leads to expressions lacking the simplicity of the residual $\partial^2 T_{ij} / \partial x_i \partial x_j$ calculated by Lighthill. In fact further progress by Lilley's approach (as will be explained shortly) can only be made by employing intuition to decide which are the 'most important' parts of the residual OS $\neq$ 0 from the point of view of jet noise.

In the present author's view, substantial as these objections are, they do not vitiate the use of Lilley's equation for the study of the effects of mean flow on jet noise. As has been pointed out more or less elegantly by Ffowcs Williams, Crow, Phillips, Doak and Lilley in a variety of contexts, the unwary initiate to aerodynamic noise starting with the papers of Lighthill needs to be repeatedly warned that writing down Lighthill's equation is not the end of the study of the subject, which in fact involves compressible unsteady nonlinear flows that are not amenable to exact calculation to this day. Advances have occurred only to the extent that intuition has been employed to suggest a cause-and-effect relation, the cause being estimated intuitively and the effect calculated according to the equations of linear acoustics by implicitly assuming that the effect does not influence the cause. A certain arbitrary splitting of a string of terms derived from the equations of motion by an equals sign followed by estimates of the terms on the right-hand side of the equality is a characteristic feature of the subject. The work of Phillips (1960) was the first major departure from the Lighthill point of view. All that Lilley (1972), Goldstein & Howes (1973) etc. have suggested is that the right-hand side of Phillips' equation is still not free of terms linear in the fluctuating quantities and that taking the material derivative of Phillips' equation serves to eliminate all terms linear in the fluctuating quantities from the right-hand side. At this point one has to draw on the selective discrepancies of the Lighthill theory/data comparison revealed by the work of Lush, Hoch, etc. to conclude that perhaps the missing element in Lighthill's work is the neglect of the effect of the mean flow on the radiation by the 'source' elements. It is difficult, having come to this conclusion, to resist linearizing the left-hand side of such an equation and drawing an analogy with stable sound propagation in a laminar parallel shear flow, the inference being that this is the dominant flow interaction effect. Of course there is no way of rigorously proving this from first principles for the reasons given by Ffowcs Williams, Crow, etc. and alluded to earlier. All that we can do is to accept the Lilley equation as one step beyond the Lighthill equation. It certainly appears to be in the right direction though how much so can be determined only by an attempt to exercise its full consequences and carry out comparisons with available data, particularly data on the aspects not explainable on the basis of Lighthill's work. To the present author at least, grappling with the equations of Lilley and Phillips and appreciating the gross nature of the approximations needed even with these equations to make a plausible and tractable 'cause-and-effect' connexion with jet noise proved an essential prerequisite to even understanding the significance of remarks such as that of Crow (1970) that the "problem of aerodynamic sound is not closed by the assertion that T_{ij} accounts for all the phenomena of compressible, rotational flow".

In the light of these remarks, it is appropriate to consider again some of the specific objections to Lilley's formulation. With regard to the unstable solutions to the OS equation, the point is that it is precisely these instabilities that have created jet turbulence and therefore one ought, in fact, deliberately to ignore such contributions in a passive-analogy model that attempts to calculate the far-field radiation (known observationally to be stable, bounded and repeatable) from sources whose qualitative features are inferred from our admittedly limited knowledge of jet turbulence. One must not however press this point too far for Ffowcs Williams (1974*a*) has pointed out that for flows of sufficiently high speed the distinction between instability waves and acoustic waves disappears and indeed there are far-field phenomena such as the 'crackling' of high-speed jets that are a direct acoustic consequence of large-scale jet instability. The present paper is not germane to such high-speed phenomena and we deliberately avoid specifying the precise jet velocity at which these occur (though experimental evidence places these as occurring at jet velocities greater than about 2·5 times the atmospheric speed of sound).

The analytical difficulties associated with solving OS $= s(\bar{x}, t)$ are avoided in the present study by the following approximation. First the inhomogeneous form of the OS equation developed by Lilley is used to identify the dominant source elements of $s(\bar{x}, t)$. Once these have been identified, it is recognized that the solution of the problem $\mathscr{L}_{OS}p = s$ with a known s characterized by low frequencies ($\mathscr{L}_{OS}$ denotes the linear Orr–Sommerfeld operator with variable coefficients) can be obtained approximately by solving the problem of noise radiation from the same source elements embedded in a plug-flow jet. We simply employ the well-known result from the acoustics of inhomogeneous media that, if the inhomogeneity occurs over a length much smaller than a wavelength, a good approximation is obtained by treating the inhomogeneity as a step-function discontinuity and employing correct matching conditions across it. Thus, in the present study the plug-flow model is used only as an analytically convenient low frequency approximation.

There is no doubt that the objection to Lilley's work that it views jet turbulence as a small perturbation superposed on a laminar parallel shear flow is valid. All one can do is to repeat the point made earlier that, inadequate as this view is, it certainly represents a valid first attempt to incorporate the influence of the flow on the radiation by the source elements. The last point concerning Lilley's equation, that it leads to complicated source terms, seems no more than a reflexion of all the difficulties of trying to deal exactly with compressible, rotational, unsteady, nonlinear flows.

In concluding the introduction, it is perhaps worth noting that Lighthill (1954, pp. 11, 12) himself was somewhat concerned as to how well the reader would accept the idea of quadrupole convection without inclusion of the effect of the jet flow itself on the radiation by the quadrupoles. He discusses the problem at some length in the cited reference and concludes by conceding that, in his model, "the quadrupoles can move but not the fluid". Since the convection of the quadrupoles is itself an effect arising from the jet flow, it appears somewhat artificial to neglect the flow of the jet fluid and yet retain eddy convection. Lighthill probably regarded his work as a valid low frequency theory (under this condition the jet flow was presumed by him to be acoustically compact and hence ignorable). But one of the most interesting conclusions of the present study is that presence of the jet flow affects both low and high frequency radiation. Thus the Lighthill analysis leading to (3) is not a valid low frequency limit. To describe the Lighthill analysis leading to (3) as a valid low jet Mach number limit appears somewhat gratuitous as it was precisely the motivation to extend his results to high jet Mach numbers that led Lighthill to develop the result (3). In any case, when hot jet flows are considered, it emerges that (3) is not a valid low Mach number result either. In fact in the case of the noise of heated jets with low velocities some fairly profound differences from the Lighthill point of view arise upon the inclusion of the effect of the jet flow. Crudely speaking, there is now a need to revise significantly one's ideas about both the right-hand and left-hand side of (1). Lilley's elegant formulation (1972) reveals both mean-flow shrouding and additional source terms due to the gradient of the mean density or temperature. Because the noise of heated jets exhibits so many unusual features, it was decided to deal with that problem separately in part 2 of the current study.

2. Formulation and method of solution

Both Lilley (1972) and Goldstein & Howes (1973) have developed equations in which, under certain restrictions, a more explicit relation than (1) can be obtained in so far as the generation of aerodynamic noise is concerned. Lilley's development is very briefly sketched below so as to indicate clearly the assumptions.

Consider an inviscid non-heat-conducting gas. The continuity, momentum and energy equations (in the absence of heat, mass and momentum sources) can be written as

$$\frac{1}{\rho}\frac{D\rho}{Dt}+\frac{\partial u_i}{\partial x_i}=0, \tag{4}$$

$$\frac{1}{\gamma}\frac{Du_i}{Dt}+\frac{a^2}{p}\frac{\partial p}{\partial x_i}=0, \tag{5}$$

$$\frac{1}{p}\frac{Dp}{Dt}-\frac{\gamma}{\rho}\frac{D\rho}{Dt}=0, \tag{6}$$

where D/Dt denotes differentiation following the fluid. Also let $r=\log(p/p_0)$, where p_0 is a reference pressure. Then, eliminating ρ from (4)–(6), we may deal with the equations

$$\frac{Dr}{Dt}+\gamma\frac{\partial u_i}{\partial x_i}=0, \tag{7}$$

$$\gamma\frac{Du_i}{Dt}+a^2\frac{\partial r}{\partial x_i}=0. \tag{8}$$

Lilley (1972) next decomposes all the field variables u_i, r and a^2 into a steady part and a fluctuating part, i.e. as $u_i=\overline{u}_i+u_i'$, etc. The mean values of (7) and (8) are next subtracted from (7) and (8) themselves. Lilley's (1972) equation follows from manipulations of (7) and (8) and three additional assumptions, namely (*a*) that the mean flow is at constant static pressure, (*b*) that whenever second-order products of fluctuating quantities such as $r'u_i'$, r'^2, $(a^2)'u_i'$, $(a^2)'r'$ and $[(a^2)']^2$ appear they may be neglected (however second-order velocity products of the form $u_i'u_i'$ are retained) and (*c*) that the mean flow is unidirectional (say $\overline{u}_i=V_1\delta_{1i}$) and varies in only one direction normal to the x_1 direction (say along the x_2 direction). Also $\overline{a^2}$ is a function only of x_2. This leads to Lilley's equation:

$$\frac{\overline{D}^3r'}{\overline{D}t^3}+2\frac{\partial V_1}{dx_2}\frac{\partial}{\partial x_1}\left[\overline{a^2}\frac{\partial r'}{\partial x_2}\right]-\frac{\overline{D}}{\overline{D}t}\frac{\partial}{\partial x_i}\left(\overline{a^2}\frac{\partial r'}{\partial x_i}\right)$$
$$=-2\gamma\frac{dV_1}{dx_2}\frac{\partial^2}{\partial x_1\partial x_k}[u_2'u_k']+\gamma\frac{\overline{D}}{\overline{D}t}\frac{\partial^2[u_i'u_j']}{\partial x_i\partial x_j}\ldots. \tag{9}$$

From this point on, Lilley's equation needs an interpretation similar to that by Lighthill of (2). The quantity r', for values of p' small compared with the ambient pressure p_A, may be shown to be equal to p'/p_A. The u_i' on the right-hand side of (9) are regarded as the known, solenoidal, turbulent velocity fluctuations and (9) then provides the required correct inhomogeneous wave equation for p'/p_A

driven by the turbulent velocity field. The improvement of (9) over (1) or even Phillips' (1960) equation is that the source term is clearly in the form of a quadratic function of the fluctuating velocities. The operator $\bar{D}/\bar{D}t$ in (9) stands for $\partial/\partial t + V_1\partial/\partial x_1$. We shall deal in the present study primarily with the noise produced by the source term

$$\gamma\frac{\bar{D}}{\bar{D}t}\frac{\partial^2}{\partial x_i\partial x_j}[u_i'u_j'].$$

Both Lilley's equation (9) and Lighthill's equation (1) exhibit 'self-noise' and 'shear-noise' source terms. However, the relationship between the self-noise term in Lilley's equation, namely

$$\frac{\bar{D}}{\bar{D}t}\frac{\partial^2}{\partial x_i\partial x_j}[u_i'u_j'],$$

and the shear-noise term in Lilley's equation, i.e.

$$\frac{dV_1}{dx_2}\frac{\partial^2}{\partial x_1\partial x_k}[u_2'u_k'],$$

is quite different from that for Lighthill's equation, where the analogous terms would be

$$\frac{\partial^2}{\partial x_i\partial x_j}(u_i'u_j'), \quad \frac{dV_1}{\partial x_2}\frac{\partial u_2'}{\partial x_1}.$$

Lighthill's equation (1) suggests the following three notions concerning shear noise and self-noise. First, it appears that shear noise might be much more important than self-noise since shear noise is only linear in the turbulent velocities while self-noise is quadratic in the turbulent velocities. This is what Lighthill (1952, 1954) had in mind when he referred to the 'amplifying' effect of mean flow gradients on jet noise. Second, it appears that the shear noise may be responsible for the low frequency sound with the self-noise accounting for the high frequency sound. Related to this is the observation by Jones (1968) that shear noise should have a convection factor of $(1-M_c\cos\theta)^{-3}$ as opposed to the $(1-M_c\cos\theta)^{-5}$ factor for self-noise. Finally, unlike self-noise, which has an isotropic or omnidirectional character, the shear-noise term exhibits a preferred axial directionality. In Lilley's formulation, the first notion is not true while the other two carry over in a somewhat weaker form. The shear-noise term in Lilley's equation is quadratic in the fluctuating velocities as is the self-noise term. The self-noise term of Lilley's equation would be characterized by a higher frequency content than the shear-noise term though this difference in frequency content appears less substantial when we note the following. The operator D/Dt operating on the self-noise term $\partial^2(u_i'u_j')/\partial x_i\partial x_j$ essentially multiplies it by ω_0, where ω_0 is a frequency of the self-noise eddy in its own (convected) frame of reference. However, the experimental work of Davies, Fisher & Barratt (1963) has shown that $\omega_0 \sim dV_1/\partial x_2$ and thus there exists a considerable qualitative similarity in frequency content between the self-noise and shear-noise terms of

Lilley's equation, though admittedly this inference leans heavily on the experimental result of Davies *et al.* Finally, it is true that the scalar function

$$\partial^2(u_2' u_k')/\partial x_1 \partial x_k$$

associated with the shear-noise term of Lilley's equation has a mildly preferred axial orientation as compared with the isotropic function $\partial^2(u_i' u_j')/\partial x_i \partial x_j$ associated with the self-noise term. The main conclusion of the above analysis is that there appears to be considerably less need to differentiate between the self-noise and shear-noise terms in (9) than, say, in (2).

In both this part and part 2, we shall deal only with solutions to (9) with a source term of the form

$$\frac{D}{Dt}\frac{\partial^2}{\partial x_i \partial x_j}(u_i' u_j'),$$

to be abbreviated as

$$\frac{D}{Dt}\left[\frac{\partial^2}{\partial x_i \partial x_j}(Q_{ij})\right],$$

where $Q_{ij} = u_i' u_j'$. In a recent study, Ffowcs Williams (1974*b*) concludes from a quite different point of view that amplification by mean-flow gradients is unlikely to occur. He shows by ingenious manipulation of the source element involving $(x_i x_j / c_0^2 |x|^2)\, \partial^2 T_{ij}/\partial t^2$ in Lighthill's equation that this can be recast in a form that reveals that only the material derivatives (D/Dt) of the T_{ij} really contribute to the noise, so that for a predominantly unidirectional flow there would be no contribution directly from transverse gradients. One further point with regard to (9) worth noting is that since the jet flow is at constant static pressure $\overline{a^2}$ can be written as $\gamma p_A/\overline{\rho(x_2)}$. Then for an r' dependence on x_1 and t of the form $\exp[i(\alpha x_1 - \omega t)]$ and r' small, the homogeneous portion of (9), i.e. (9) with the right-hand side set equal to zero, yields that across a thin shear layer the quantity

$$\frac{1}{\overline{\rho(x_2)}\,(\omega - \alpha V_1(x_2))^2}\left(\frac{\partial p'}{\partial x_2}\right)$$

must be continuous. This is of course equivalent to the usual kinematic condition that the transverse acoustic particle displacement be continuous across the shear layer. The reason for pointing out this feature of the homogeneous form of Lilley's equation (9) is that the homogeneous form of Phillips' (1960) equation fails to yield the correct kinematic condition when examined in the limit of a vanishingly thin shear layer.

As indicated in the introduction, in what follows we shall deal with (9) in a cylindrical co-ordinate system (see figure 1) where V_1 = constant for $0 \leqslant r < a$ and V_1 = zero otherwise. The source term is associated with solenoidal turbulent velocity fluctuations and hence a suitable choice of a fundamental form for Q_{ij} would be $Q_{ij}^0 \delta(y)\,\delta(z)\,\delta(x - V_c t)\exp(i\omega_0 t)$ (where Q_{ij}^0 is a constant), which would represent an eddy embedded in the jet, convecting along its centre-line at V_c. Centre-line eddy convection would be representative of an average result for eddies distributed across the jet cross-section. For every eddy located

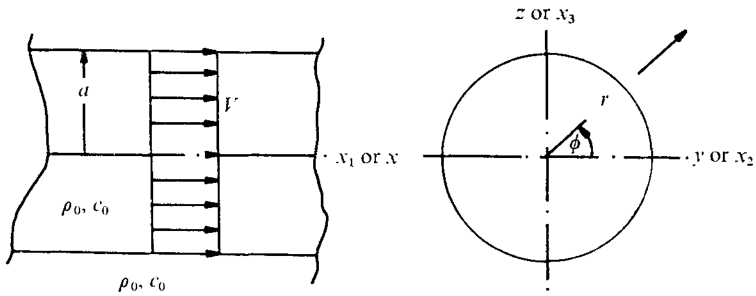

FIGURE 1

off the axis there is an uncorrelated eddy located symmetrically at the image of the first eddy in the jet axis. Some simplified calculations with plane plug-flow jet models in Mani (1974) indicate that the average of the results of two uncorrelated sources located symmetrically about the jet axis (but still lying within the jet) is well represented by the field of one source placed on the axis. A similar experimental result for a low velocity jet flow may be seen in figure 16 of Atvars *et al.* (1966), where it is shown that the average sound field of two sources placed symmetrically off the axis (along the nozzle lip line) is very closely approximated by the sound field of one source placed on the jet centre-line. It was these results of Atvars *et al.* (1966) that were used by Schubert (1969) in his extensive numerical study of jet refraction to justify his use of sources placed on the jet centre-line. However, in a recent paper, Ffowcs Williams (1974*b*) has suggested that the high frequency noise emitted from regions of the mixing layer close to the nozzle exit plane ought to be modelled by convected sources placed just outside a plane vortex sheet separating the ambient fluid from a region of uniform flow. In this paper the speed of sound in the jet and its density are assumed to have the same values c_0 and ρ_0 as the ambient fluid. ω_0 would represent the oscillation frequency of the eddy in its own frame of reference. The mathematical problem then involves solving

$$\frac{\bar{D}}{\bar{D}t}(\nabla^2 p') - \frac{1}{c_0^2}\frac{\bar{D}^3 p'}{\bar{D}t^3} = -\rho_0 Q_{ij}^0 \frac{\bar{D}}{\bar{D}t}\frac{\partial^2}{\partial x_i \partial x_j}[\delta(y)\,\delta(z)\exp(i\omega_0 t)\,\delta(x - V_c t)]$$
$$\text{inside the jet} \quad (0 \leqslant r < a), \tag{10}$$

where $$\bar{D}/\bar{D}t = \partial/\partial t + V_1 \partial/\partial x,$$

and $$\nabla^2 p' - \frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} = 0 \quad \text{outside the jet} \quad (r > a). \tag{11}$$

Across the jet/still-air interface p' and the transverse acoustic particle displacement η should be continuous. The relation between p' and η is

$$\frac{\bar{D}^2\eta}{\bar{D}t^2} = \frac{-1}{\rho_0}\frac{\partial p'}{\partial r} \quad \text{inside the jet} \tag{12}$$

and $$\frac{\partial^2\eta}{\partial t^2} = \frac{-1}{\rho_0}\frac{\partial p'}{\partial r} \quad \text{outside the jet.} \tag{13}$$

We now perform Fourier transformation of (10)–(13) parallel to the x axis. In doing so we note that the transform of $\delta(x-V_c t)$ is $\exp(i\alpha V_c t)$ (where α is the Fourier transform variable). Since $\exp(i\omega_0 t)\,\delta(x-V_c t)$ is the driving term of the whole problem, this means that the x,t dependence of all quantities in the Fourier-transform formulation will be $\exp(-i\alpha x)\exp[i(\omega_0+\alpha V_c)t]$. This implies the following operator equivalence in (10)–(13):

$$\partial/\partial t \sim i(\omega_0+\alpha V_c), \quad \partial^2/\partial x^2 \sim -\alpha^2, \quad \bar{D}/\bar{D}t \sim i(\omega_0+\alpha V_c-\alpha V_1).$$

In what follows it is convenient to assume that $V_1 = V_c = V$. Such an assumption is consistent with a plug-flow model for the jet flow. This assumption is not necessary but it does help to simplify the algebra. Also one can see that four types of basic quadrupole solution to (10)–(13) need to be worked out, corresponding to the x-x, x-y, y-y and y-z types of quadrupole.

Consider first the x-x case. We may rewrite (10)–(13) in the form of a problem in the transverse (y,z) co-ordinates:

$$\nabla^2_{y,z}p' + (k_0^2-\alpha^2)P' = \rho_0 Q^0_{xx}\alpha^2\delta(y)\,\delta(z) \quad \text{for} \quad 0 \leqslant r < a$$

$$\text{(note that } \nabla^2_{y,z} \text{ stands for } \partial^2/\partial y^2+\partial^2/\partial z^2\text{)}, \qquad (14)$$

$$\nabla^2_{y,z}P' + [(k_0+\alpha M)^2-\alpha^2]P' = 0 \quad \text{for} \quad r > a, \qquad (15)$$

$$N = \frac{1}{\rho_0\omega_0^2}\frac{\partial P'}{\partial r} \quad \text{for} \quad 0 \leqslant r < a \qquad (16)$$

and

$$N = \frac{1}{\rho_0\omega_0^2(1+\alpha M/k_0)^2}\frac{\partial P'}{\partial r} \quad \text{for} \quad r > a. \qquad (17)$$

Note that $k_0 = \omega_0/c_0$, $M = V/c_0$ and that P' and N are the axial Fourier transforms of p' and η. Since we are interested only in propagating solutions in the far field, the range of α of interest is $-k_0/(1+M) \leqslant \alpha \leqslant k_0/(1-M)$. We are primarily interested in the solution for P' for $r > a$ and this works out in the case of (14)–(17) to be

$$P'(\alpha) = -\rho_0 Q^0_{xx}\alpha^2 H_0^{(2)}(\alpha^+ r)\Big/ 2\pi\left\{H_0^{(2)}(\alpha^+ a)\,J_0'(\tilde{\alpha}^+ a)\,(\tilde{\alpha}^+ a) - \frac{J_0(\tilde{\alpha}^+ a)\,H_0^{(2)\prime}(\alpha^+ a)\,(\alpha^+ a)}{(1+\alpha M/k_0)^2}\right\}$$

$$\text{for} \quad -\frac{k_0}{1+M} \leqslant \alpha \leqslant k_0 \qquad (18)$$

and

$$P'(\alpha) = -\rho_0 Q^0_{xx}\alpha^2 H_0^{(2)}(\alpha^+ r)\Big/ 2\pi\left\{(\hat{\alpha}^+ a)\,I_0'(\hat{\alpha}^+ a)\,H_0^{(2)}(\alpha^+ a) - \frac{(\alpha^+ a)\,H_0^{(2)\prime}(\alpha^+ a)\,I_0(\hat{\alpha}^+ a)}{(1+\alpha M/k_0)^2}\right\}$$

$$\text{for} \quad k_0 \leqslant \alpha \leqslant \frac{k_0}{1-M}, \qquad (19)$$

where α^+ is the positive square root $[(k_0+\alpha M)^2-\alpha^2]^{\frac{1}{2}}$, $\tilde{\alpha}^+$ the positive square root $(k_0^2-\alpha^2)^{\frac{1}{2}}$ and $\hat{\alpha}^+$ the positive square root $(\alpha^2-k_0^2)^{\frac{1}{2}}$. Thus p' can be written as

$$p' = \frac{1}{2\pi}\int_{-\infty}^{\infty} P'(\alpha)\exp[-i\alpha(x-Vt)]\exp(i\omega_0 t)\,d\alpha. \qquad (20)$$

In the notation of figure 2, with $x-Vt = R'\cos\theta'$ and $r = R'\sin\theta'$ we may

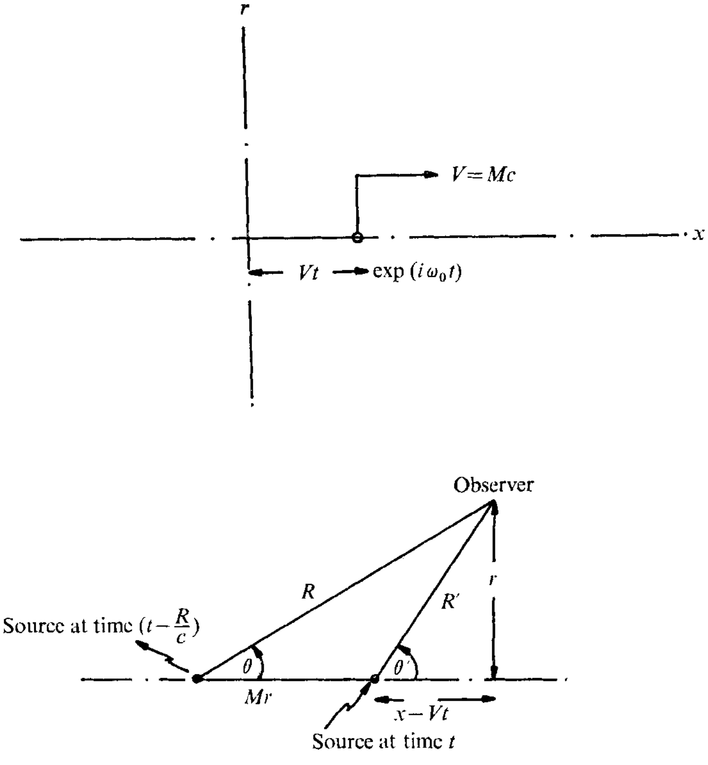

FIGURE 2

examine (20) by the method of stationary phase for large R'. The details are omitted but we may show that with retarded co-ordinates (R, θ) defined as in figure 2 and with $P'(\alpha)$ written as $F(\alpha)\, H_0^{(2)}(\alpha^+ r)$, the far-field expression for p' is

$$p' \sim \frac{iF(\alpha_0)\exp(i\omega_0 t)\exp(-ik_0 R)}{\pi R(1 - M\cos\theta)}, \tag{21}$$

where

$$\alpha_0 = k_0 \cos\theta/(1 - M\cos\theta). \tag{22}$$

It is worth noting that $\alpha^+(\alpha_0) = k_0 \sin\theta/(1 - M\cos\theta)$. The result (21) ignores the Kelvin–Helmholtz instability associated with the jet/still-air interface. In a recent study (Mani 1974), the reason why such a step is permissible was discussed in some detail. Basically the argument is that such instabilities have in fact created the jet turbulence and hence should not be included again when analysing the radiation due to this turbulence (in this calculation we are assuming the turbulent source terms to be known). The real jet flow represents a statistically stable system and the plug-flow model is merely an artifice employed in the analysis to assess conveniently the mean-flow shrouding effects.

The Doppler-shift formula is obtained by noting that the time rate of variation of the phase $\omega_0 t - k_0 R$ in (21) is $\omega_0/(1 - M\cos\theta)$ (note that R varies with time). Thus the expression for the far-field pressure due to an x-x quadrupole can be written as

$$\frac{-i\rho_0 Q^0_{xx}\omega_0^2\cos^2\theta\exp[i(\omega_0 t-k_0R)]}{2\pi^2Rc_0^2(1-M\cos\theta)^3\{\hat{\alpha}^+aI_0'(\hat{\alpha}^+a)H_0^{(2)}(\alpha^+a)-(\alpha^+a)(1-M\cos\theta)^2H_0^{(2)\prime}(\alpha^+a)I_0(\hat{\alpha}^+a)\}}$$

for $0\leqslant\theta\leqslant\cos^{-1}[(1+M)^{-1}]$, where $\hat{\alpha}^+$ and α^+ are to be evaluated for

$$\alpha=k_0\cos\theta/(1-M\cos\theta),$$

and for $\cos^{-1}[(1+M)^{-1}]\leqslant\theta\leqslant\pi$ as

$$p'=\frac{-i\rho_0 Q^0_{xx}\omega_0^2\cos^2\theta\exp[i(\omega_0 t-k_0R)]}{2\pi^2Rc_0^2(1-M\cos\theta)^3\{(\tilde{\alpha}^+a)J_0'(\tilde{\alpha}^+a)H_0^{(2)}(\alpha^+a)-(\alpha^+a)(1-M\cos\theta)^2H_0^{(2)\prime}(\alpha^+a)J_0(\tilde{\alpha}^+a)\}}. \tag{23}$$

Expression (23) differs in many respects from the corresponding Lighthill expression for a freely convected quadrupole. If the expression corresponding to (23) had been derived keeping track of the difference between V_1 and V_c, the Lighthill expression would be the limit of that expression as $V_1\to 0$ with non-zero

$$V_c\;(=c_0M_c),$$

i.e.

$$p'\sim\frac{-\rho_0 Q^0_{xx}\omega_0^2\cos^2\theta\exp[i(\omega_0 t-k_0R)]}{4\pi Rc_0^2(1-M_c\cos\theta)^3}. \tag{24}$$

This is of course hardly surprising since Lighthill (1952, 1954) himself stated that his model was one in which the eddy moved (i.e. V_c was arbitrary) but the fluid did not (i.e. $V_1=0$).

The major difference between (23) and (24) is that in the former the far-field directivity is completely frequency dependent. The relevant non-dimensional parameters governing the directivity are now M and k_0a. For

$$0\leqslant\theta\leqslant\cos^{-1}[(1+M)^{-1}]$$

(the so-called 'zone of silence') and high k_0a the exponential nature of the I functions is a manifestation of refraction of the sound by the jet. Also, for non-zero k_0a, $p'\to 0$ logarithmically as $\theta\to 0$ or $\theta\to\pi$. (Gottlieb (1960) refers to this as the 'Lloyd's mirror' effect.) A most interesting result is obtained by examining (23) as $k_0a\to 0$ (low frequency result): we find that

$$p'\sim\frac{-\rho_0 Q^0_{xx}\omega_0^2\cos^2\theta\exp[i(\omega_0 t-k_0R)]}{4\pi c_0^2R(1-M\cos\theta)^5}. \tag{25}$$

(If the problem were worked with $V_1\neq V_c$, the expression $(1-M\cos\theta)^5$ in the denominator of (25) would be modified to $(1-M_c\cos\theta)^3(1-M_1\cos\theta)^2$, where $M_c=V_c/c_0$ and $M_1=V_1/c_0$.) In other words (24) is not a valid low frequency limit (as was mentioned in the introduction). Such a feature of low frequency noise emission was first noticed experimentally by Mollo-Christensen & Narasimha (1960) and qualitatively ascribed by them to the influence of the jet flow. Berman (1974) has also drawn attention to it, pointing out that it is not an instability effect but rather "the noise generation process is enhanced by a fully stable resonance phenomenon".

Equations (25) and (23) to some extent explain why an expression of the form (24), due to Lighthill, has seemed to work well in the past, at least for the noise of cold jets. It turns out, roughly speaking, that (24) underestimates the variation of p' with respect to θ at low frequencies [when compared with (23)], as is indicated by (25), while overestimating it at high frequencies. The overestimation arises essentially because, as pointed by Ribner (1960, 1962), Powell (1960) and Csanady (1966), at high frequencies the radiation of an eddy is primarily governed by its own immediate environment (namely the jet flow), with respect to which it is not convecting at all. Regardless of how high $k_0 a$ may be, both $\alpha^+ a$ and $\hat{\alpha}^+ a$ or $\tilde{\alpha}^+ a$ approach zero as $\theta \rightarrow 0$ or π and as $\theta \rightarrow \cos^{-1}[(1+M)^{-1}]$. Because of this, it was not possible to extract a high frequency limit of (23). Besides, a plug-flow model of the jet flow is obviously a poor model at high frequencies. In any event, this feature of underestimation of noise generation at low frequencies by (24) and overestimation at high frequencies is apparently the reason why expressions of this type essentially succeeded in explaining jet-noise directivity in the past when such directivities were measured for the overall sound pressure (i.e. the integral of the pressure spectrum over all frequencies). Jet-noise research owes a great deal to the selective plotting of directivities at constant source frequencies (i.e. at constant $k_0 a$) in the manner of Lush (1971), Ahuja & Bushell (1973) and others. It is such displays that have revealed clearly the shortcomings of expressions of the form of (24). We should like to conclude this discussion of (23) by emphasizing that it is an expression that exhibits simultaneously the combined convection–refraction effect, which is so crucial to the determination of jet-noise directivity. It also emphasizes the need to plot all jet-noise directivity data at constant source frequencies as this is the only directivity plot that can be checked directly against an acoustic theory. It is the only manner in which we can bypass our current inability to predict the turbulence source spectrum in detail.

Turning now to the sound fields of quadrupoles of x-y type, the transverse nature of these singularities turns out to impose a basic difference between the way these sound fields were deduced by Lighthill (1952, 1954) and the manner in which they must be deduced from (10)–(13). The left-hand side of Lighthill's equation consists only of constant-coefficient operators (namely $\partial^2/\partial t^2 - a_0^2 \nabla^2$). Also the only boundary condition associated with (1) is a 'radiation' condition. This means that having solved (1) with a right-hand side of the form $\delta(x)\,\delta(y)\,\delta(z)$, say, one may differentiate this solution in *the far field* with respect to x_i and x_j to obtain the results for the higher-order singularities. In other words the well-known reciprocity of the solutions to (1) with respect to the observer and source positions enables one to derive the higher-order singular solutions in the far field with relative ease. Equations (10)–(13) or Lilley's equations are however homogeneous only in time and in the axial direction (assuming a non-spreading jet) and thus only singularities involving derivatives of a source term with respect to time or the axial co-ordinate (as is the case for the x-x quadrupole) allow such a simple solution procedure. Equations (10)–(13) are inhomogeneous

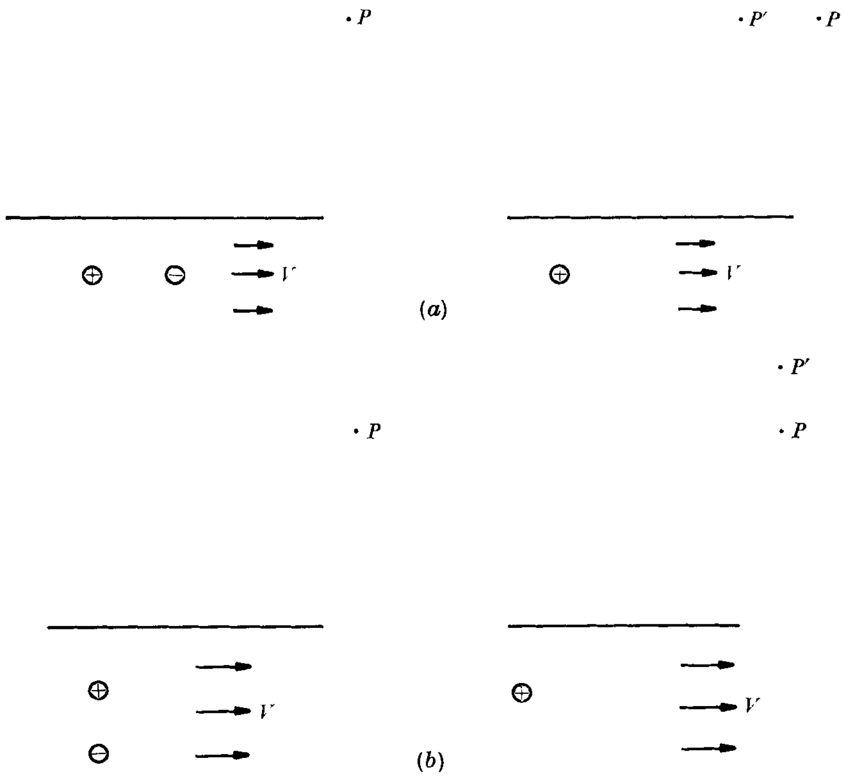

FIGURE 3. (*a*) To derive the solution at P due to the sources on the left, we may take the difference between the solutions at P and P' due to one source as shown on the right. (*b*) A similar equivalence does not hold for this case.

with regard to the transverse co-ordinates y and z and hence special procedures must be adopted to derive the sound fields of the transverse singularities. Figure 3 is an attempt to illustrate this difference between Lighthill's and Lilley's formulation.

Two approaches may be employed to deal with this situation. The first, more general approach is to derive the fundamental solutions for an arbitrary transverse source position in the jet. This means solving for a fundamental form

$$\delta(x - Vt)\exp(i\omega_0 t)\,\delta(y - y_0)\,\delta(z - z_0).$$

It is now perfectly admissible to differentiate the far-field form of this solution with regard to the source co-ordinates (y_0, z_0) (and *then*, if so desired, set y_0, $z_0 = 0$) to derive the higher-order singular solutions. This is because the coefficients and boundary conditions associated with (10)–(13) *are* independent of the source position, i.e. (y_0, z_0). The only problem with this approach is that for arbitrary (y_0, z_0) even the first fundamental solution will be non-axisymmetric (owing to the asymmetric source location in the transverse plane). For this reason a second, more restricted approach will be adopted in the following.

Specializing to the case of an x-y quadrupole, the problem in the y, z plane for the transforms P' and N for (10)–(13) is

$$\nabla^2_{y,z}P' - \alpha^2 P' + k_0^2 P' = +i\alpha\rho_0 Q^0_{xy}\,\partial[\delta(y)\,\delta(z)]/\partial y \quad \text{for} \quad 0 \leqslant r \leqslant \alpha, \tag{26}$$

$$\nabla^2_{y,z}P' + [(k_0+\alpha M)^2 - \alpha^2]\,P' = 0 \quad \text{for} \quad r > a, \tag{27}$$

$$N = \frac{1}{\rho_0 \omega_0^2}\frac{\partial P'}{\partial r} \quad \text{for} \quad 0 \leqslant r < a \tag{28}$$

and

$$N = \frac{1}{\rho_0 \omega_0^2 (1+\alpha M/k_0)^2}\frac{\partial P'}{\partial r} \quad \text{for} \quad r > a. \tag{29}$$

The solution for P' is (for $r > a$)

$$P' = +i\alpha\rho_0 Q^0_{xy}\hat{\alpha}^+ \cos\phi H_1^{(2)}(\alpha^+ r) \Big/ 2\pi\Big\{H_1^{(2)}(\alpha^+ a)\,(\hat{\alpha}^+ a)\,I_1'(\hat{\alpha}^+ a) - \frac{(\alpha^+ a)\,H_1^{(2)\prime}(\alpha^+ a)\,I_1(\hat{\alpha}^+ a)}{(1+\alpha M/k_0)^2}\Big\}$$

for $k_0 \leqslant \alpha \leqslant k_0/(1-M)$ and

$$P' = +i\alpha Q^0_{xy}\tilde{\alpha}^+ \cos\phi H_1^{(2)}(\tilde{\alpha}^+ r) \Big/ 2\pi\Big\{H_1(\alpha^+ a)\,(\hat{\alpha}^+ a)\,J_1'(\tilde{\alpha}^+ a) - \frac{(\alpha^+ a)\,H_1^{(2)\prime}(\alpha^+ a)\,J_1(\tilde{\alpha}^+ a)}{(1+\alpha M/k_0)^2}\Big\} \tag{30}$$

for $-k_0/(1+M) \leqslant \alpha \leqslant k_0$. As before, if we write $P' = F(\alpha)\,H_1^{(2)}(\alpha^+ r)$, then in the far field p' is given by

$$p' \sim \frac{-F(\alpha_0)\exp[i(\omega_0 t - k_0 R)]}{\pi R(1-M\cos\theta)},$$

with $\alpha_0 = k_0\cos\theta/(1-M\cos\theta)$. Explicitly, then, for the x-y quadrupole

$$p' \sim \frac{-i\rho_0\omega_0^2(\alpha^+/k_0)\cos\theta\cos\phi\exp[i(\omega_0 t - k_0 R)]\,Q^0_{xy}}{2\pi^2 R c_0^2 (1-M\cos\theta)^2\{(\hat{\alpha}^+ a)\,I_1'(\hat{\alpha}^+ a)\,H_1^{(2)}(\hat{\alpha}^+ a) - (\alpha^+ a)\,(1-M\cos\theta)^2 H_1^{(2)\prime}(\alpha^+ a)\,I_1(\hat{\alpha}^+ a)\}}$$

for $0 \leqslant \theta \leqslant \cos^{-1}[(1+M)^{-1}]$, where $\alpha^+ = k_0\sin\theta/(1-M\cos\theta)$ and $\hat{\alpha}^+ = (\alpha^2 - k_0^2)^{\frac{1}{2}}$ is to be evaluated for $\alpha = k_0\cos\theta/(1-M\cos\theta)$. For $\cos^{-1}[(1+M)^{-1}] \leqslant \theta \leqslant \pi$,

$$p' \sim \frac{-i\rho_0\omega_0^2(\tilde{\alpha}^+/k_0)\cos\theta\cos\phi\exp[i(\omega_0 t - k_0 R)]\,Q^0_{xy}}{2\pi^2 R c_0^2 (1-M\cos\theta)^2\{(\tilde{\alpha}^+ a)\,J_1'(\tilde{\alpha}^+ a)\,H_1^{(2)}(\alpha^+ a) - (\alpha^+ a)\,(1-M\cos\theta)^2 H_1^{(2)\prime}(\alpha^+ a)\,J_1(\tilde{\alpha}^+ a)\}}. \tag{31}$$

Equation (31), in addition to the frequency dependence of (23), exhibits another significant difference from the Lighthill result for an x-y quadrupole. This is the fact that the explicit convection amplification factor appearing in (31) is only $(1-M\cos\theta)^{-2}$ and not $(1-M\cos\theta)^{-3}$. An additional (frequency-independent) convection factor is contained in $\hat{\alpha}^+/k_0$ or $\tilde{\alpha}^+/k_0$ and works out to be

$$|1 - 2M\cos\theta - \cos^2\theta(1-M^2)|^{\frac{1}{2}}/(1-M\cos\theta).$$

The above expression, in general (especially for $0 \leqslant \theta \leqslant \pi_2$), is less than

$$(1-M\cos\theta)^{-1}.$$

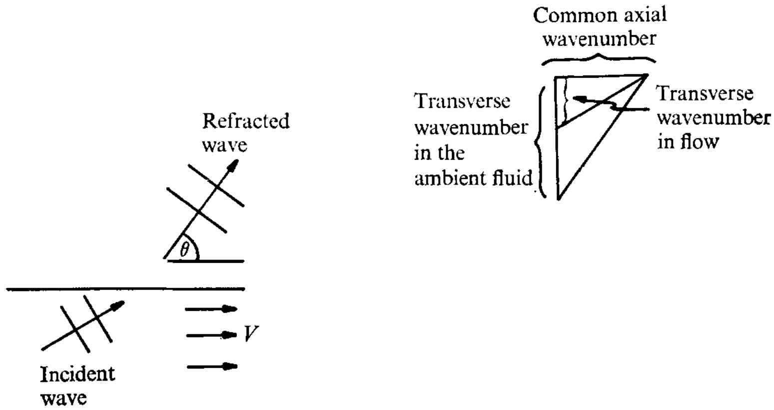

FIGURE 4. Phase speed or wavenumber differences between incident and refracted waves.

This is due to the fact that the enhancement of the phase cancellation (which is responsible for the convective amplification) for the transverse component of the singularity is now related inversely to the transverse phase speed of the sound wave *within* the flow. As shown in figure 4 for a plane wave, this phase speed is (for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$) greater than that of the refracted wave outside the flow. Such a feature need not be considered for an axial quadrupole because the axial phase speeds are matched inside and outside the flow in figure 4.

The theory for the y-y and y-z quadrupoles can be worked out analogously and we give only the final results.

For $0 \leqslant \theta \leqslant \cos^{-1}[(1+M)^{-1}]$, with the usual definitions of α^+ and $\hat{\alpha}^+$, for the y-y quadrupole

$$p' \sim \frac{i\rho_0 Q^0_{yy}\exp[i(\omega_0 t - k_0 R)]\,\omega_0^2(\hat{\alpha}^+/k_0)^2}{4\pi^2 R(1-M\cos\theta)\,c_0^2} \times\left\{\frac{-\cos 2\phi}{[(\hat{\alpha}^+a)\,I_2'(\hat{\alpha}^+a)\,H_2^{(2)}(\alpha^+a)-(\alpha^+a)\,(1-M\cos\theta)^2\,H_2^{(2)\prime}(\alpha^+a)\,I_2(\hat{\alpha}^+a)]} + \frac{1}{[(\hat{\alpha}^+a)\,I_0'(\hat{\alpha}^+a)\,H_0^{(2)}(\alpha^+a)-(\alpha^+a)\,(1-M\cos\theta)^2\,H_0^{(2)\prime}(\alpha^+a)\,I_0(\hat{\alpha}^+a)]}\right\}. \quad (32)$$

For $\cos^{-1}[(1+M)^{-1}] \leqslant \theta \leqslant \pi$, the same expression applies except that $\hat{\alpha}^+$ should be replaced by $\tilde{\alpha}^+$ and the I's by J's. For the y-z quadrupole, for

$$0 \leqslant \theta \leqslant \cos^{-1}[(1+M)^{-1}]$$

$$p' \sim \frac{-i\rho_0 Q^0_{yz}\,\omega_0^2(\hat{\alpha}^+/k_0)^2\exp[i(\omega_0 t - k_0 R)]\sin 2\phi}{4\pi^2 c_0^2 R(1-M\cos\theta)} \times\left\{\frac{1}{(\hat{\alpha}^+a)\,I_2'(\hat{\alpha}^+a)\,H_2^{(2)}(\alpha^+a)-(1-M\cos\theta)^2\,(\alpha^+a)\,H_0^{(2)\prime}(\alpha^+a)\,I_2(\hat{\alpha}^+a)}\right\}. \quad (33)$$

For $\cos^{-1}[(1+M)^{-1}] \leqslant \theta \leqslant \pi$, a similar expression applies with $\hat{\alpha}^+$ replaced by $\tilde{\alpha}^+$ and the I's by J's.

From (32) and (33), we see that the only explicit convection factor that appears is now $(1-M\cos\theta)^{-1}$. In fact this term appears from the stationary-phase method of evaluation of integrals of the form of (20) and represents the purely volumetric dilatation effect due to singularity convection identified by Lighthill (1952, 1954). When evaluating the intensity, i.e. $\langle p'^2 \rangle$, this term will be corrected from $(1-M\cos\theta)^{-2}$ to $(1-M\cos\theta)^{-1}$ to allow for the reduced number of quadrupoles emitting simultaneously (Ffowcs Williams 1963). There are additional convection effects associated with the term $(\hat{\alpha}^+/k_0)^2$ or $(\tilde{\alpha}^+/k_0)^2$ but as with the x-y quadrupole these terms are now less than the factor $(1-M\cos\theta)^{-2}$ for $0 \leqslant \theta < \frac{1}{2}\pi$.

We should like to conclude this section by pointing out two implications of these results particularly relevant to experimental work. Lighthill's equation (1) has tremendous simplicity and being essentially devoid of any boundary conditions (except the radiation condition) has the simple consequence that the directionality of the sound field for any convected quadrupole can be written in the far field as $D(\theta,\phi)(1-M_c\cos\theta)^{-3}$, where $D(\theta,\phi)$ is the intrinsic directionality of the stationary quadrupole, which is just $x_i x_j/R^2$ for an i,j quadrupole. As we have shown in this section such a simple result is invalid when one considers mean-flow shrouding effects. One consequence of Lighthill's result (for *his* equation) was that so far as the far field was concerned the source term

$$\rho_0 \frac{\partial^2}{\partial x_i \partial x_j}(u_i u_j)$$

could be replaced by

$$\frac{\rho_0}{c_0^2}\frac{\partial^2}{\partial t^2} v_r^2,$$

where v_r was the component of u_i or u_j along the line joining the source point and the observer. This result is sometimes referred to as the Lighthill–Proudman form of the source term (Ribner 1969). It has recently gained some popularity in the area of source location in jets by cross-correlation of an in-jet measurement taken with an instrument such as an optical or hot-wire anemometer (used to measure v_r^2) with the output of a far-field microphone (Lee & Ribner 1972). Such an identification of $\rho_0 \partial^2 v_r^2/\partial t^2$ as a source term is valid only to the extent that Lighthill's equation is valid. It is not a valid source-location procedure if we admit the importance of mean-flow shrouding effects and hence the need to pose the noise generation problem in terms of convected wave equations.

Second, the term

$$(\tilde{\alpha}^+ a)\, J_n'(\tilde{\alpha}^+ a)\, H_n^{(2)}(\alpha^+ a) - (1-M\cos\theta)^2\, (\alpha^+ a)\, J_n(\tilde{\alpha}^+ a)\, H_n^{(2)\prime}(\alpha^+ a)\},$$

which appears in the denominators of (23), (31), (32) and (33) with $n = 0, 1$ or 2 for $\cos^{-1}[(1+M)^{-1}] \leqslant \theta \leqslant \pi$, degenerates into a simple constant $(2i/\pi)$ when $\theta = \frac{1}{2}\pi$ (when $\alpha = 0$ and $\tilde{\alpha}^+ = \alpha^+$). This is related to the obvious result that, for cold jets, one does not expect any mean-flow shrouding effects at the 90° location (there being no component of flow along this direction). Thus Lilley's equation (for cold jets) yields results at $\theta = 90°$ identical to those of Lighthill's equation. Experimentally then, at $\theta = 90°$ one does expect (for cold jets) the scaling principle deduced from Lighthill's equation to work. This means that at $\theta = 90°$ one should get good 8 power-law scaling of the intensity and good Strouhal scaling with regard to jet velocity, nozzle size, etc. This is in fact what recent experiments on cold jets by Lush (1971) and Ahuja & Bushell (1973) confirm, namely that the Lighthill theory does work well at $\theta = 90^\circ$.

3. Application of theory to experimental results

In a recent study, Ribner (1969) has explained how the fundamental solutions associated with the various quadrupoles can be employed to derive the axially symmetric sound field of a round jet. The results for the contribution due to 'self-noise' in his study (which uses Lighthill's equation (1) and the associated Lighthill–Proudman form of the source function as a basis) are particularly relevant here. Ribner (1969) studied the expression for the mean-square pressure, and by employing a model of homogeneous isotropic turbulence in their own frame of reference for the eddies and by examining the directional average with respect to the ϕ co-ordinate of the sound field, was able to ascribe 'weights' to the various quadrupole contributions. Essentially, the six basic quadrupoles (x-x, x-y, x-z, y-z, y-y and z-z) contribute independently though there are weak cross-quadrupole contributions (i.e. of type x-x, y-y, x-x, z-z, y-y, z-z, etc.). With some slight liberties, we may derive from Ribner's study the conclusion that the self-noise contribution may be evaluated from the formula

far-field intensity $\sim$ (mean-square pressure of x-x quadrupole)
$+ 4 \times$ (circumferential average of mean-square pressure of x-y or x-z quadrupoles)
$+ 2 \times$ (circumferential average of mean-square pressure of y-y or z-z quadrupoles) $+ 2 \times$ (circumferential average of mean-square pressure of y-z quadrupole). (34)

The only difference between the above formula and that of Ribner is the neglect of the weak cross-quadrupole contributions (i.e. of the x-x, y-y, x-x, z-z and y, y, z, z types). Both the above formula and Ribner's more exact result yield a basic omnidirectional pattern for the self-noise for Lighthill's equation except for the $(1 - M_c \cos\theta)^{-5}$ convection effect. The other assumptions used in the comparison with the data are as follows.

(*a*) To allow for jet spread, etc., the comparisons are all carried out assuming eddies moving at 65% of the nominal ideal-jet exit velocity in a plug flow, which is itself assumed to be 65% of the nominal ideal-jet exit velocity. As noted earlier, with a plug-flow jet model, it seemed inconsistent to allow 'slip' between the eddy convection velocity and the jet velocity and this assumption seemed to be the best compromise. It should be noted that the assumption of eddies convecting at 65% of the normal ideal-jet exit velocity is one that is commonly used by experimentalists (Lush 1971; Ahuja-Bushell 1973, etc.). We also assume centre-line eddy convection to be representative of the average result for eddies distributed across the cross-section.

(*b*) The predictions in this part are made by assuming $(\rho_j, c_j) = (\rho_0, c_0)$ (jet density and speed of sound equal to those of the ambient fluid). The principal quantity predicted is the directional distribution of the sound pressure intensity for fixed values of the nominal jet exit velocity and source frequency parameter $k_0 a$. This involves combining the results for the various quadrupoles according to the formula indicated earlier. As in all jet-noise work, we assume that the angle θ in figure 2 (the angle measured from the position of the eddy at the time of emission of the acoustical signal reaching the observer at the current time) can be identified with the angle of measurement from the jet axis quoted by the experimentalists. This assumes that the radius of measurement is large compared with the length parallel to the jet axis over which the eddies may be assumed to have a coherent existence.

(*c*) As noted in the previous section, the theory of the present paper is basically an acoustic theory and hence no attempt is made to predict the turbulence source-function spectrum. Thus only relative directional distributions are predicted. Hence, in comparisons with the data of Lush (1971) and Ahuja & Bushell (1973), one vertical adjustment of the directional distribution to 'best fit' the data has been carried out. For this reason, in figures 5–10 no absolute levels are shown though the 10 dB increment is clearly indicated. Sometimes it is suggested that such comparisons ought to be carried out by anchoring the theory and data at the 90° point. This suggestion seems to have at least two deficiencies. First (especially at high velocities and low frequencies), the pressure levels at the 90° point are as much as 20 dB below the peak value. It seems unwise to anchor the prediction to a location where the pressure levels are orders of magnitude weaker than at the location of peak intensity. Second, most theories of jet noise deal with doubly infinite jet columns, thereby failing to account for the presence or influence of the tail-pipe on the directivity of the noise. Inclusion of tail-pipe effects would give rise to mixed boundary-value problems. Thus we should not expect such theories to yield very accurate predictions for

$$\tfrac{1}{2}\pi \leqslant \theta \leqslant \pi$$

(in this angular sector one would expect some influence of the tail-pipe).

(*d*) Finally, as in Lighthill's theory, we assume the convected sources to be compact.

With these assumptions, in figures 5–10 computed directivity patterns at constant source frequencies for jet Mach numbers ranging from about 0·35 to 0·91 are compared with the data of Lush (1971) and Ahuja & Bushell (1973). Source Strouhal numbers ranging from 0·03 to 1·0 were covered in these two sets of experiments and results for four of these are compared with the current theory.

In general the agreement between theory and experiment appears to be very good. The two sets of experiments give data at very comparable conditions and it is extremely difficult to pinpoint instances where the theory fails systematically with both sets of data. The reduced convective amplification of the transverse quadrupoles (see for instance figure 9 between $\theta = 60°$ and $\theta = 90°$, where the radiation is dominated by the transverse quadrupoles), the reduced amplification at high frequencies and the balance between convection and refraction are all correctly predicted and apparent in the data. The reader may refer to both

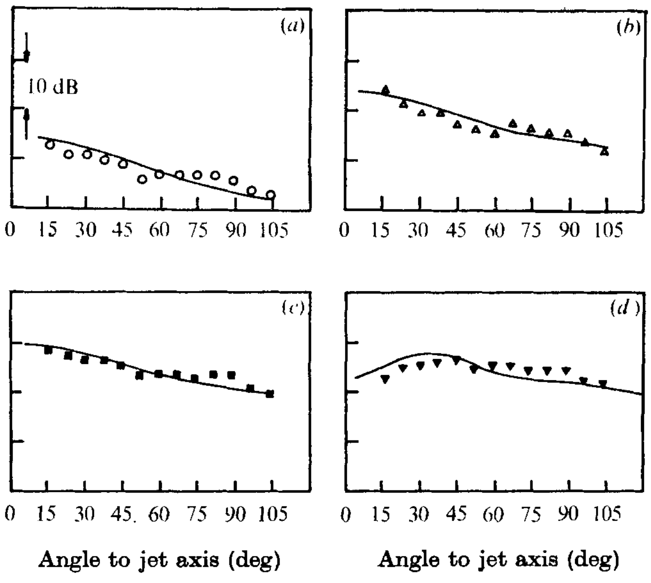

FIGURE 5. Comparison with data of Lush for $M_j = 0{\cdot}366$ ($V_j = 125$ m/s). ——, present theory. Source Strouhal number: (*a*) 0·03, (*b*) 0·10, (*c*) 0·30, (*d*) 1·0.

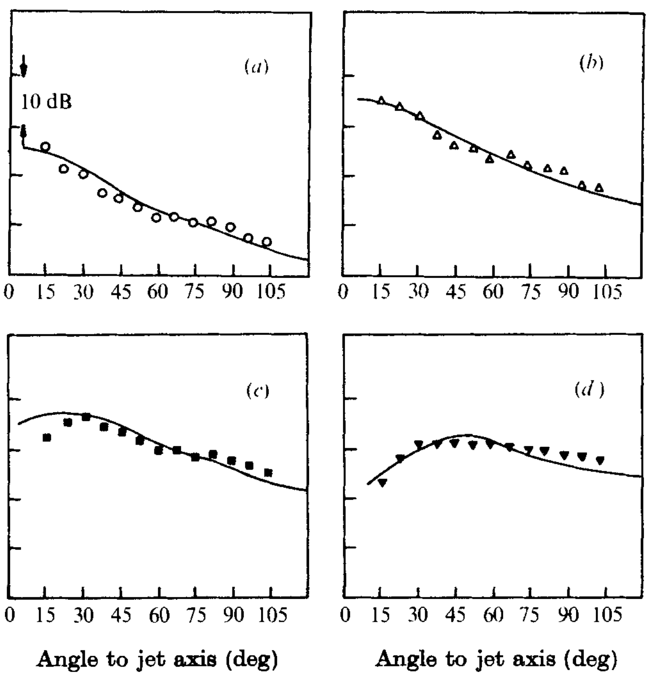

FIGURE 6. Comparison with data of Lush for $M_j = 0{\cdot}57$ ($V_j = 195$ m/s). ——, present theory. Source Strouhal number: (*a*) 0·03, (*b*) 0·10, (*c*) 0·30, (*d*) 1·0.

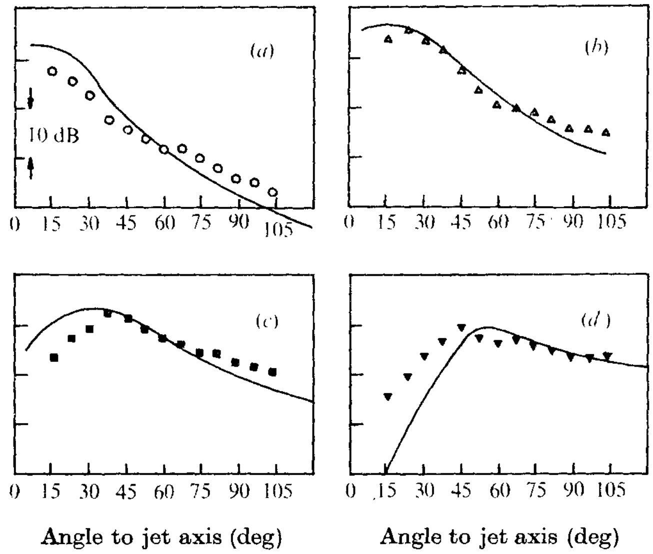

FIGURE 7. Comparison with data of Lush for $M_j = 0{\cdot}878$ ($V_j = 300$ m/s). ——, present theory. Source Strouhal number: (a) 0·03, (b) 0·10, (c) 0·30, (d) 1·0.

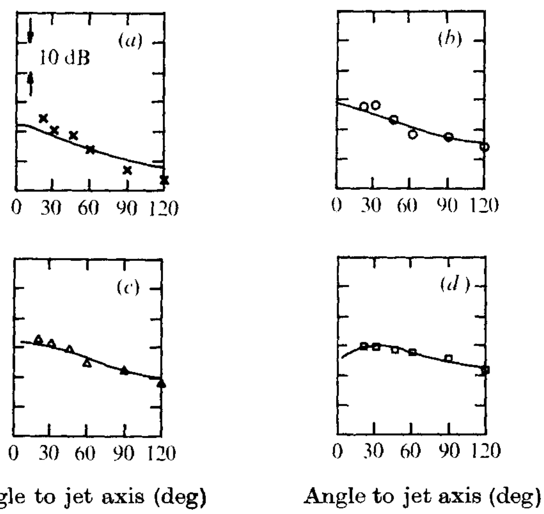

FIGURE 8. Comparison with data of Ahuja & Bushell for $M_j = 0{\cdot}363$ ($V_j = 400$ ft/s). ——, present theory. Source Strouhal number: (a) 0·03, (b) 0·10, (c) 0·30, (d) 1·0.

Lush (1971) and Ahuja & Bushell (1973) to see how well the Lighthill theory, leading to a frequency-independent directivity factor $(1-M_c\cos\theta)^{-5}$, was able to correlate the data. A systematic underestimation of the variation with θ at low frequencies and a systematic overestimation at the high frequencies by the Lighthill expression is evident. The refractive effect, of course, is not included in the Lighthill theory at all.

The only systematic deficiency found is (see figures 7 and 10) the overestimation

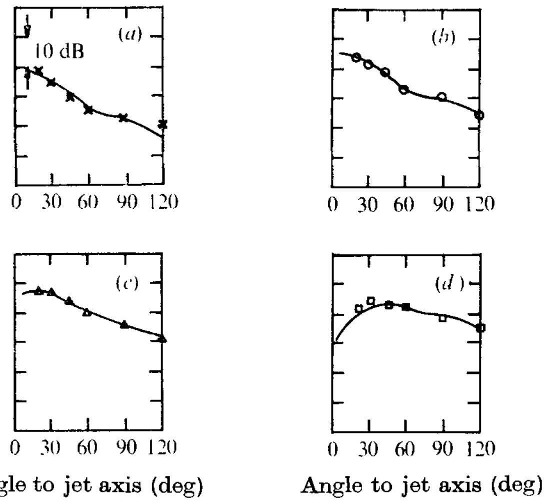

FIGURE 9. Comparison with data of Ahuja & Bushell for $M_j = 0{\cdot}546$ ($V_j = 600$ ft/s). ——, present theory. Source Strouhal number: (*a*) 0·03, (*b*) 0·10, (*c*) 0·30, (*d*) 1·0.

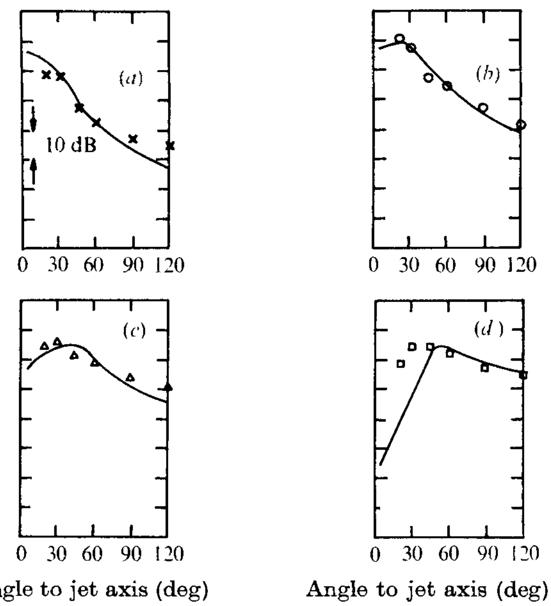

FIGURE 10. Comparison with data of Ahuja & Bushell for $M_j = 0{\cdot}909$ ($V_j = 1000$ ft/s). ——, present theory. Source Strouhal number: (*a*) 0·03, (*b*) 0·10, (*c*) 0·30, (*d*) 1·0.

of the refractive dip at the highest velocity (300 m/s and 1000 ft/s) and highest source Strouhal number (1·0) covered in these experiments. This is obviously a deficiency of the plug-flow model. It is possible that some of this discrepancy could be resolved by the suggestion of Ffowcs Williams (1974*b*) that high source Strouhal number emission is modelled better by the use of sources placed just outside a vortex sheet separating a semi-infinite region of uniform flow from a stationary ambient half-space. Even in this one instance the radiation aft of the peak is quite reasonably predicted by the current model.

As mentioned earlier, the studies of Lush and Ahuja & Bushell along with other studies have confirmed that good Strouhal number scaling with respect to velocity and nozzle size and a good eighth-power law for intensity *vs.* jet velocity are obtained at $\theta = 90^\circ$ for cold jets. This means (since we know both analytically and physically that mean-flow shrouding effects for cold jets are absent at $\theta = 90^\circ$) that the intrinsic distribution of quadrupoles generating jet noise does follow simple dimensional scaling. That is, the u_i' and u_j' do scale with V_j, ω_0 does vary as V_j/D and there does exist a universal strength distribution of Q^0/V_j^2 against $\omega_0 D/V_j$. Most important, the success of the Lighthill theory at $\theta = 90^\circ$ indicates very little (if any) compressibility effect on this distribution at least up to sonic jet velocities. The calculations of the present paper give procedures for computing the directional distribution of the sound (say in decibels relative to the 90° point) for various source frequency parameters (which can then be translated into observed frequencies using the Doppler-shift formula $(1-M_c\cos\theta)^{-1}$). Unfortunately these distributions are not simply expressible as a frequency-independent function $(1-M_c\cos\theta)^{-5}$. With the aid of a digital computer, however, calculations of the type leading to figures 5–10 can be executed with extreme rapidity. Except for this need to program the results, it is true to say that this part of the study has essentially shown that there is a frequency-dependent procedure for scaling cold-jet noise with respect to the angle from the jet axis, the nozzle size and the jet velocity. The calculations (with the assumptions outlined earlier) can be formally carried out up to nominal jet velocities of $c_0/0{\cdot}65$ (corresponding to subsonic eddy convection velocities). Agreement with experimental data has so far been shown to be good up to jet velocities around sonic. It is likely that the plug-flow model will not be directly usable at nominal jet velocities higher than about $c_0/0{\cdot}65$. In addition, procedures for dealing with the singularity at $M_c\cos\theta = 1$ will need to be devised (Ffowcs Williams 1963).

4. Concluding remarks

In the present paper we have attempted to account systematically for the effect of the mean flow on the radiation from subsonically convected quadrupoles oscillating at some frequency in their own frame of reference. Lilley's (1972) equation was used to apply the results to the problem of cold-jet noise. In the interest of deriving closed-form analytical solutions and illuminating the physics, a plug-flow model of the jet was adopted. Within this framework, one can obtain an exact representation of the balance between convective and refractive effects in jet noise. Several novel aspects of the jet-noise problem not discernible at all from the Lighthill acoustic-analogy approach have emerged from the current study.

(i) The Lighthill result for the directivity, namely the expression

$$(1-M_c\cos\theta)^{-5},$$

emerges only as the limit for zero jet flow Mach number and non-zero (arbitrary) eddy convection Mach number. It is not a good low frequency approximation. Indeed the results herein show that, fortuitously, it is some sort of approximation for the variation with angle of the overall sound pressure level. This is because of its tendency to underestimate the variation with angle of the low frequency sound while overestimating this for the high frequency sound.

(ii) When mean-flow shrouding effects are included, the technique employed by Lighthill to derive solutions for the higher-order singularities is useful only for the purely axial singularities, and even that application is possible only if the jet flow is assumed to be homogeneous in the axial co-ordinate (non-spreading jet). Inhomogeneity of the flow in the transverse direction necessitates special procedures for the development of solutions for the transverse singularities. A refracted wave emerging from the jet flow into the ambient fluid is characterized in the forward arc by a higher phase speed in the transverse direction within the flow than outside it. This results in reduced convective amplification for the transverse singularities as compared with the predictions of the Lighthill theory. Also the frequently employed equivalence of source terms

$$\frac{\partial^2}{\partial x_i \partial x_j}(u_i u_j), \quad \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}(v_r^2)$$

(Lighthill–Proudman form) is no longer valid.

(iii) Extensive comparisons of the theory (with clearly stated assumptions) with experiments of Lush (1971) and Ahuja & Bushell (1973) have been carried out. The comparisons are for the published directivity plots at constant source frequencies for jet velocities from about 100 m/s to nearly sonic velocities. The agreement in general is very good especially concerning the new insight provided by the current analysis. We refer here to the tendency of the variation with angle of the low frequency sound to exceed that predicted by the $(1-M_c\cos\theta)^{-5}$ formula and vice versa for the high frequency sound. We also refer to the tendency of the data to exhibit reduced convective amplification between $\theta = 60°$ and $\theta = 90°$ as compared with that between $\theta = 15°$ and $\theta = 60°$. This is related to the reduced convective amplification associated with the transverse quadrupoles.

(iv) The data of Lush (1971) and Ahuja & Bushell (1973) for the intensity at $\theta = 90°$ scale very well on an eighth-power law and those for the frequency on fD/V_j (Strouhal scaling). This suggests very little effect of the Mach number (or compressibility) on the turbulence source spectrum or the 'intrinsic quadrupole distribution' (at least up to sonic jet velocities). Together with the ability of the present calculations to predict the variation with angle of the intensity (at various source frequencies), it can be concluded that the problem of scaling cold-jet noise with respect to the angle from the jet axis, the jet velocity and the jet nozzle size has been essentially solved (within the limitation of subsonic eddy convection velocities).

(v) Finally it should be reiterated that several ideas first advanced by Lighthill (1952, 1954) have, in fact, been retained. The notion of convected compact eddies of quadrupole character radiating noise owing to oscillations of quadrupole strength in their own frame of reference has been adopted in its entirety from Lighthill's work. Analytically, Phillips, Lilley and others have pursued Lighthill's goal of deriving an inhomogeneous wave equation for the fluctuating pressure driven by spatial gradients of the solenoidal turbulent velocity fluctuations. The principal differences from Lighthill's point of view are accounting for the effect of the jet flow on the radiation by the eddies and the associated abandonment of the attempt to cast the problem in terms of an analogy with stationary-media acoustics.

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